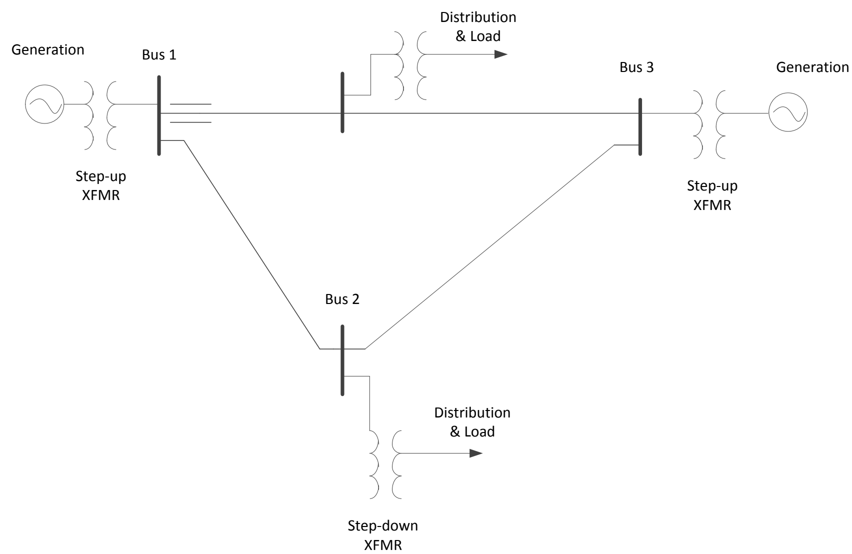
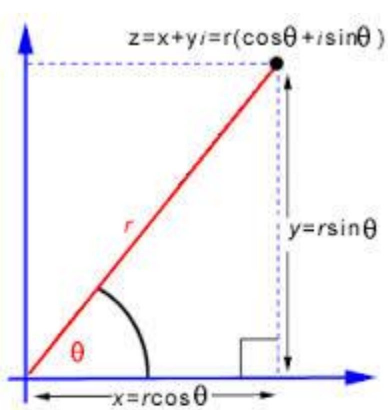
1. Plot of a simple three-node power system network (AC) system



Major quantities: Frequency (f = 60 Hz), Voltages, Currents and Flows within Transmission Network.

We will introduce a convenient form for the above quantities: Phasor. To learn Phasor, we need to first review some knowledge of Complex Numbers

2. Complex Number(s):



An arbitrary complex number $Z = x + j \cdot y = r(\cos\theta + j\sin\theta)$ with $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2 + y^2}$, and $\theta = \arctan\dfrac{y}{x}$ (Note: "*Z*" here is just the name of a complex number, please do not confuse this *Z* with impedance *Z*)

Here, j is the imaginary unit. We have $j \cdot j = j^2 = -1$. Note, in most math books, the imaginary unit is denoted as i. However, in our power system area, we use j instead of i because i has been widely used as a quantity for a current.

Euler's Identity:

$$Z = r(\cos\theta + j \cdot \sin\theta) \triangleq r \cdot e^{j\theta}$$

where $\triangleq$ means "being defined as." In other words, whenever we see $e^{j \cdot \theta}$, we should interpret that it is a mathematical notation for $(\cos\theta + j \cdot \sin\theta)$. All features of $e^{j \cdot \theta}$ are actually derived from $(\cos\theta + j \cdot \sin\theta)$.

$Z = x + j \cdot y$ is called "rectangular form"

$Z = r(\cos\theta + j\sin\theta)$ is called "polar form" or "modulus-and-argument form" or "absolute-value and-argument form."

3. Arithmetic of Complex Numbers

For $Z_1 = x_1 + j \cdot y_1$ and $Z_2 = x_2 + j \cdot y_2$, we have

Addition: $Z_1 + Z_2 = (x_1 + x_2) + j \cdot (y_1 + y_2)$

Subtraction: $Z_1 - Z_2 = (x_1 - x_2) + j \cdot (y_1 - y_2)$

Multiplication: $Z_1 \cdot Z_2 = (x_1 + j \cdot y_1) \times (x_2 + j \cdot y_2)$

$$= (x_1 x_2 + j^2 y_1 y_2) + j \cdot (x_1 y_2 + x_2 y_1)$$
$$= (x_1 x_2 - y_1 y_2) + j \cdot (x_1 y_2 + x_2 y_1)$$

or $Z_1 \cdot Z_2 = r_1 \cdot r_2 \cdot e^{j \cdot (\theta_1 + \theta_2)}$

Division: $\frac{Z_1}{Z_2} = (x_1 + j \cdot y_1)/(x_2 + j \cdot y_2)$

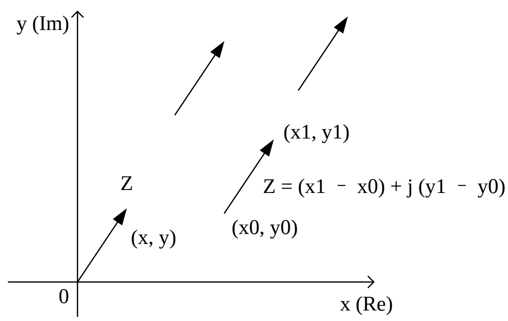
$$= (x_1 + j \cdot y_1) \times (x_2 - j \cdot y_2)/(x_2^2 + y_2^2)$$

or $\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \cdot e^{j \cdot (\theta_1 - \theta_2)}$

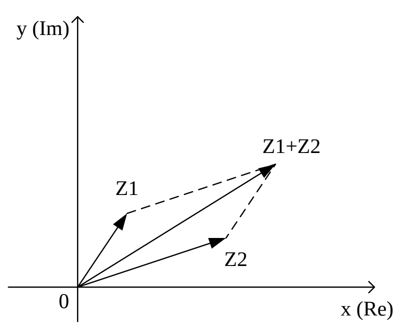
Our Observations: For operations of addition and subtraction, the rectangular form is simpler while for operations of multiplication and division, the polar form is simpler.

4. Illustration of Complex Number Arithmetic

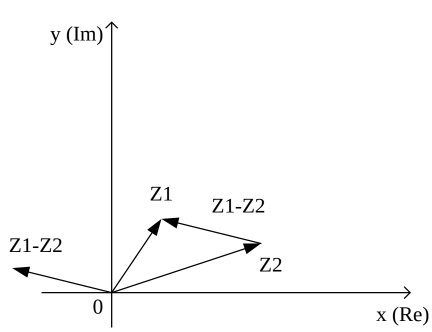
4.1 Generalized vector form of complex number on Complex Plane



4.2 Addition of Two Complex Numbers:



4.3 Subtraction of Complex Numbers:



4.4 Multiplication and Division – Rotation(s):

r1*r2 and Θ1+Θ2 for multiplication (and r1/r2 and Θ1-Θ2)

5. Phasor for an AC (Alternating Circuit) signal

The following derivation and proof is given for the purpose of further understanding only (not required):

For two arbitrary AC voltage (or current) signals, they should have the following waveforms:

$v_1(t) = \sqrt{2}|V_1|\sin(\omega t + \theta_{v,1})$ and $v_2(t) = \sqrt{2}|V_2|\sin(\omega t + \theta_{v,2})$.

Applying Euler's Equation,

$$v_1(t) = \sqrt{2}|V_1|\sin(\omega t + \theta_{v,1}) = \mathrm{Im}\{\sqrt{2}|V_1|\cos(\omega t + \theta_{v,1}) + j\cdot\sqrt{2}|V_1|\sin(\omega t + \theta_{v,1})\}$$

$$= \mathrm{Im}\{\sqrt{2}|V_1|e^{j(\omega t + \theta_{v,1})}\} = \mathrm{Im}\{\sqrt{2}|V_1|e^{j\theta_{v,1}}e^{j\omega t}\}$$

and similarly,

$$v_2(t) = \mathrm{Im}\{\sqrt{2}|V_2|e^{j(\omega t + \theta_{v,2})}\} = \mathrm{Im}\{\sqrt{2}|V_2|e^{j\theta_{v,2}}e^{j\omega t}\}$$

Here, Im{} means "taking the imaginary part of." Then

$$v_1(t) - v_2(t) = \mathrm{Im}\{\sqrt{2}|V_1|e^{j\theta_{v,1}}e^{j\omega t}\} - \mathrm{Im}\{\sqrt{2}|V_2|e^{j\theta_{v,2}}e^{j\omega t}\}$$

$$= \mathrm{Im}\{\sqrt{2}|V_1|e^{j\theta_{v,1}}e^{j\omega t} - \sqrt{2}|V_2|e^{j\theta_{v,2}}e^{j\omega t}\}$$

$$= \mathrm{Im}\{\sqrt{2}\left(|V_1|e^{j\theta_{v,1}} - |V_2|e^{j\theta_{v,2}}\right)e^{j\omega t}\}$$

Clearly, $v_1(t) \equiv v_2(t)$ or $v_1(t) - v_2(t) = 0$ holds true for all time t **if and only if** $|V_1|e^{j\theta_{v,1}} - |V_2|e^{j\theta_{v,2}} = 0$, that is, $|V_1|e^{j\theta_{v,1}} = |V_2|e^{j\theta_{v,2}}$. These two complex numbers are the phasors of $v_1(t)$ and $v_2(t)$, respectively. End

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In other words, an AC voltage signal with a given frequency can be completely represented by its phasor: Once we know $v(t) = \sqrt{2}|V|\sin(\omega t + \theta_v)$, we can uniquely determine that its phasor is $V = |V|e^{j\cdot\theta_v}$; Once we know an AC signal's phasor $V = |V|e^{j\cdot\theta_v}$, then we can uniquely determine that the signal's waveform is .

6. Why use Phasors? – Simplifying the calculation for AC circuits

Resistor: $v_R(t) = R \cdot i_R(t) \quad \Leftrightarrow \quad V_R = R \cdot I_R$

Inductor: $v_L(t) = L \cdot \frac{di_L(t)}{dt} \quad \Leftrightarrow \quad V_L = j\omega L \cdot I_L$

Capacitor: $i_c(t) = C \cdot \frac{dv_c(t)}{t} \quad \Leftrightarrow \quad I_c = j\omega C \cdot V_c \quad \Leftrightarrow \quad V_c = \frac{1}{j\omega C} \cdot I_c$

All differential operations disappeared. Resistors, Inductors, and Capacitors are represented as (Complex) Impedances now.

Kirchhoff's Laws:

Voltage Law: $\sum v_m(t) = 0 \quad \Leftrightarrow \quad \sum V_m = 0$

Current Law: $\sum i_n(t) = 0 \quad \Leftrightarrow \quad \sum I_n = 0$

Conclusions:

I. The Phasor of an AC voltage (or current) signal can completely represent the original time-varying signal.

II. Using Phasors, AC circuit analysis in power systems becomes as simple as DC circuit analysis with complex numbers.

Remember:

$$\cos(\omega t + \theta) = \sin(\omega t + \theta + 90^\circ) \text{ in degree}$$

$$= \sin(\omega t + \theta + \frac{\pi}{2}) \text{ in radian}$$