Power Flow Solutions

Power Flow Equations

In the system shown at right line models to have a resistance of 0.01pu in addition to a series reactance of 0.1pu.

- All branches have same series impedance:

  \[ pu := 1 \]
  \[ Z_{\text{series}} := 0.01 + j \cdot 0.1 \text{pu} \]
  \[ Y_{\text{series}} := \frac{1}{Z_{\text{series}}} \]
  \[ Y_{\text{series}} = (0.99 - 9.901i) \text{pu} \]

- All branches have same shunt admittance (note that this is the value at each end, so it has already been divided by 2).

  \[ Z_{\text{shunt}} := -j \cdot 10 \text{pu} \]
  \[ Y_{\text{shunt}} := \frac{1}{Z_{\text{shunt}}} \]
  \[ Y_{\text{shunt}} = 0.1i \text{pu} \]

Matrix elements:

\[ y_{11} := 3 \cdot Y_{\text{series}} + 3 \cdot Y_{\text{shunt}} \]
\[ y_{33} := 2 \cdot Y_{\text{series}} + 2 \cdot Y_{\text{shunt}} \]
\[ y_{22} := 2 \cdot Y_{\text{series}} + 2 \cdot Y_{\text{shunt}} \]
\[ y_{44} := 3 \cdot Y_{\text{series}} + 3 \cdot Y_{\text{shunt}} \]

\[
Y_{\text{bus}} := \begin{pmatrix}
  y_{11} & -Y_{\text{series}} & -Y_{\text{series}} & -Y_{\text{series}} \\
  -Y_{\text{series}} & y_{22} & 0 & -Y_{\text{series}} \\
  -Y_{\text{series}} & 0 & y_{33} & -Y_{\text{series}} \\
  -Y_{\text{series}} & -Y_{\text{series}} & -Y_{\text{series}} & y_{44}
\end{pmatrix}
\]

\[
Y_{\text{bus}} := \begin{pmatrix}
  2.97 - 29.403i & -0.99 + 9.901i & -0.99 + 9.901i & -0.99 + 9.901i \\
  -0.99 + 9.901i & 1.98 - 19.602i & 0 & -0.99 + 9.901i \\
  -0.99 + 9.901i & 0 & 1.98 - 19.602i & -0.99 + 9.901i \\
  -0.99 + 9.901i & -0.99 + 9.901i & -0.99 + 9.901i & 2.97 - 29.403i
\end{pmatrix} \text{pu}
\]
Reset origin for matrix and vector subscripts from 0 to 1. \( \text{ORIGIN} := 1 \)

Complex Form Equations:

\[
\begin{align*}
S_1 &= V_1 \cdot V_1' \cdot Y_{bus_{1,1}} + V_1 \cdot V_2' \cdot Y_{bus_{1,2}} + V_1 \cdot V_3' \cdot Y_{bus_{1,3}} + V_1 \cdot V_4' \cdot Y_{bus_{1,4}} \\
S_2 &= V_2 \cdot V_1' \cdot Y_{bus_{2,1}} + V_2 \cdot V_2' \cdot Y_{bus_{2,2}} + V_2 \cdot V_3' \cdot Y_{bus_{2,3}} \\
S_3 &= V_3 \cdot V_1' \cdot Y_{bus_{3,1}} + V_3 \cdot V_2' \cdot Y_{bus_{3,2}} + V_3 \cdot V_3' \cdot Y_{bus_{3,3}} + V_3 \cdot V_4' \cdot Y_{bus_{3,4}} \\
S_4 &= V_4 \cdot V_1' \cdot Y_{bus_{4,1}} + V_4 \cdot V_2' \cdot Y_{bus_{4,2}} + V_4 \cdot V_3' \cdot Y_{bus_{4,3}} + V_4 \cdot V_4' \cdot Y_{bus_{4,4}}
\end{align*}
\]

Find real and imaginary parts of the \( Y_{bus} \) matrix:

\[ G := \text{Re}(Y_{bus}) \]
\[
G = \begin{bmatrix}
2.97 & -0.99 & -0.99 & -0.99 \\
-0.99 & 1.98 & 0 & -0.99 \\
-0.99 & 0 & 1.98 & -0.99 \\
-0.99 & -0.99 & -0.99 & 2.97
\end{bmatrix}
\]

\[ B := \text{Im}(Y_{bus}) \]
\[
B = \begin{bmatrix}
-29.403 & 9.901 & 9.901 & 9.901 \\
9.901 & -19.602 & 0 & 9.901 \\
9.901 & 0 & -19.602 & 9.901 \\
9.901 & 9.901 & 9.901 & -29.403
\end{bmatrix}
\]
Rectangular Form

\[ P_1 = (|V_1|^2) \cdot G_{1,1} + |V_1| \cdot |V_2| \cdot (G_{1,2} \cdot \cos(\theta_1 - \theta_2) + B_{1,2} \cdot \sin(\theta_1 - \theta_2)) + |V_1| \cdot |V_3| \cdot (G_{1,3} \cdot \cos(\theta_1 - \theta_3) + B_{1,3} \cdot \sin(\theta_1 - \theta_3)) + |V_1| \cdot |V_4| \cdot (G_{1,4} \cdot \cos(\theta_1 - \theta_4) + B_{1,4} \cdot \sin(\theta_1 - \theta_4)) \]

\[ P_2 = |V_2| \cdot |V_1| \cdot (G_{2,1} \cdot \cos(\theta_2 - \theta_1) + B_{2,1} \cdot \sin(\theta_2 - \theta_1)) + (|V_2|^2) \cdot G_{2,2} + |V_2| \cdot |V_4| \cdot (G_{2,4} \cdot \cos(\theta_2 - \theta_4) + B_{2,4} \cdot \sin(\theta_2 - \theta_4)) \]

\[ P_3 = |V_3| \cdot |V_1| \cdot (G_{3,1} \cdot \cos(\theta_3 - \theta_1) + B_{3,1} \cdot \sin(\theta_3 - \theta_1)) + (|V_3|^2) \cdot G_{3,3} + |V_3| \cdot |V_4| \cdot (G_{3,4} \cdot \cos(\theta_3 - \theta_4) + B_{3,4} \cdot \sin(\theta_3 - \theta_4)) \]

\[ P_4 = |V_4| \cdot |V_1| \cdot (G_{4,1} \cdot \cos(\theta_4 - \theta_1) + B_{4,1} \cdot \sin(\theta_4 - \theta_1)) + |V_4| \cdot |V_2| \cdot (G_{4,2} \cdot \cos(\theta_4 - \theta_2) + B_{4,2} \cdot \sin(\theta_4 - \theta_2)) + |V_4| \cdot |V_3| \cdot (G_{4,3} \cdot \cos(\theta_4 - \theta_3) + B_{4,3} \cdot \sin(\theta_4 - \theta_3)) + (|V_4|^2) \cdot G_{4,4} \]

\[ Q_1 = -(|V_1|^2) \cdot B_{1,1} + |V_1| \cdot |V_2| \cdot (G_{1,2} \cdot \sin(\theta_1 - \theta_2) - B_{1,2} \cdot \cos(\theta_1 - \theta_2)) + |V_1| \cdot |V_3| \cdot (G_{1,3} \cdot \sin(\theta_1 - \theta_3) - B_{1,3} \cdot \cos(\theta_1 - \theta_3)) + |V_1| \cdot |V_4| \cdot (G_{1,4} \cdot \sin(\theta_1 - \theta_4) - B_{1,4} \cdot \cos(\theta_1 - \theta_4)) \]

\[ Q_2 = |V_2| \cdot |V_1| \cdot (G_{2,1} \cdot \sin(\theta_2 - \theta_1) - B_{2,1} \cdot \cos(\theta_2 - \theta_1)) - (|V_2|^2) \cdot B_{2,2} + |V_2| \cdot |V_4| \cdot (G_{2,4} \cdot \sin(\theta_2 - \theta_4) - B_{2,4} \cdot \cos(\theta_2 - \theta_4)) \]

\[ Q_3 = |V_3| \cdot |V_1| \cdot (G_{3,1} \cdot \sin(\theta_3 - \theta_1) - B_{3,1} \cdot \cos(\theta_3 - \theta_1)) - (|V_3|^2) \cdot B_{3,3} + |V_3| \cdot |V_4| \cdot (G_{3,4} \cdot \sin(\theta_3 - \theta_4) - B_{3,4} \cdot \cos(\theta_3 - \theta_4)) \]

\[ Q_4 = |V_4| \cdot |V_1| \cdot (G_{4,1} \cdot \sin(\theta_4 - \theta_1) - B_{4,1} \cdot \cos(\theta_4 - \theta_1)) + |V_4| \cdot |V_2| \cdot (G_{4,2} \cdot \sin(\theta_4 - \theta_2) - B_{4,2} \cdot \cos(\theta_4 - \theta_2)) + |V_4| \cdot |V_3| \cdot (G_{4,3} \cdot \sin(\theta_4 - \theta_3) - B_{4,3} \cdot \cos(\theta_4 - \theta_3)) - (|V_4|^2) \cdot B_{4,4} \]

You could plug the numbers for $G_{ij}$ and $B_{ij}$ into the equations next.
Newton-Raphson Example 1:

Reset origin for matrix and vector subscripts from 1 back to 0. ORIGIN := 0

Use a simpler \( Y_{bus} \) matrix:

\[
Y_{bus} := \begin{pmatrix}
-j19.98 & j10 & j10 \\
-j19.98 & j10 & j10 \\
j10 & j10 & -j19.98 \\
\end{pmatrix}
\]

\[\begin{align*}
P_2 &= 0.5 \\
P_3 &= -1 \\
Q_2 &= 0.5 \\
Q_3 &= -1 \\
\end{align*}\]

Slack Bus: \(V_{1m} := 1.0\) \(\theta_1 := 0\)

Initial Guesses:

\[\begin{align*}
V_{2m0} &= 1.0 \\
\theta_{20} &= 0 \\
V_{3m0} &= 1.0 \\
\theta_{30} &= 0 \\
\end{align*}\]

Power flow equations using initial guess:

\[\begin{align*}
P_{20} &= V_{2m0} - V_{1m} \cdot \text{Im}(Y_{bus_{1,0}}) \cdot \sin(\theta_{20} - \theta_1) + V_{2m0} \cdot V_{3m0} \cdot \text{Im}(Y_{bus_{1,2}}) \cdot \sin(\theta_{20} - \theta_{30}) \\
P_{30} &= V_{3m0} - V_{1m} \cdot \text{Im}(Y_{bus_{2,0}}) \cdot \sin(\theta_{30} - \theta_1) + V_{3m0} \cdot V_{2m0} \cdot \text{Im}(Y_{bus_{2,1}}) \cdot \sin(\theta_{30} - \theta_{20}) \\
Q_{20} &= -V_{2m0} \cdot V_{1m} \cdot \text{Im}(Y_{bus_{1,0}}) \cdot \cos(\theta_{20} - \theta_1) - V_{2m0}^2 \cdot \text{Im}(Y_{bus_{1,1}}) - V_{2m0} \cdot V_{3m0} \cdot \text{Im}(Y_{bus_{1,2}}) \cdot \cos(\theta_{20} - \theta_{30}) \\
Q_{30} &= -V_{3m0} \cdot V_{1m} \cdot \text{Im}(Y_{bus_{2,0}}) \cdot \cos(\theta_{30} - \theta_1) - V_{3m0} \cdot V_{2m0} \cdot \text{Im}(Y_{bus_{2,1}}) \cdot \cos(\theta_{30} - \theta_{20}) - V_{3m0}^2 \cdot \text{Im}(Y_{bus_{2,2}}) \\
\end{align*}\]

Initial Mismatch Vector:

\[
\Delta H_0 := \begin{pmatrix}
(P_2 - P_{20}) \\
(P_3 - P_{30}) \\
(Q_2 - Q_{20}) \\
(Q_3 - Q_{30}) \\
\end{pmatrix}
\]

\[\Delta H_0 = \begin{pmatrix}
0.5 \\
-1 \\
0.52 \\
(-0.98) \\
\end{pmatrix}\]

Use a "1-norm"

\[\text{norm}_1 := \begin{cases} 
\text{out} \leftarrow 0 & \text{for } x \in 0..3 \\
\text{out} \leftarrow \text{out} + |\Delta H_0|_x & \text{well out of tolerance} 
\end{cases}\]

\[\text{norm}_1 = 3\]
Jacobian Terms

J11 submatrix
\[
\begin{align*}
J_{P28_0} & := \text{Im}(Y_{bus_{1,0}}) \cdot V_{2m0} \cdot \text{V1m} \cdot \cos(\theta_{20} - \theta_{11}) + \text{Im}(Y_{bus_{1,2}}) \cdot V_{2m0} \cdot \text{V3m0} \cdot \cos(\theta_{20} - \theta_{30}) \\
J_{P28_3} & := -\text{Im}(Y_{bus_{1,2}}) \cdot V_{2m0} \cdot \text{V3m0} \cdot \cos(\theta_{20} - \theta_{30}) \\
J_{P38_2} & := -\text{Im}(Y_{bus_{2,1}}) \cdot V_{3m0} \cdot \text{V2m0} \cdot \cos(\theta_{30} - \theta_{20}) \\
J_{P38_3} & := \text{Im}(Y_{bus_{2,0}}) \cdot V_{3m0} \cdot \text{V1m} \cdot \cos(\theta_{30} - \theta_{11}) + \text{Im}(Y_{bus_{2,1}}) \cdot V_{3m0} \cdot \text{V2m0} \cdot \cos(\theta_{30} - \theta_{20})
\end{align*}
\]

J12 submatrix
\[
\begin{align*}
J_{P2V2_0} & := \text{Im}(Y_{bus_{1,0}}) \cdot \text{V1m} \cdot \sin(\theta_{20} - \theta_{11}) + \text{Im}(Y_{bus_{1,2}}) \cdot V_{3m0} \cdot \sin(\theta_{20} - \theta_{30}) \\
J_{P2V3_0} & := \text{Im}(Y_{bus_{1,2}}) \cdot V_{2m0} \cdot \sin(\theta_{20} - \theta_{30}) \\
J_{P3V2_0} & := \text{Im}(Y_{bus_{2,1}}) \cdot V_{3m0} \cdot \sin(\theta_{30} - \theta_{20}) \\
J_{P3V3_0} & := \text{Im}(Y_{bus_{2,0}}) \cdot \text{V1m} \cdot \sin(\theta_{30} - \theta_{11}) + \text{Im}(Y_{bus_{2,1}}) \cdot V_{2m0} \cdot \sin(\theta_{30} - \theta_{20})
\end{align*}
\]

J21 submatrix
\[
\begin{align*}
J_{Q28_0} & := \text{Im}(Y_{bus_{1,0}}) \cdot V_{2m0} \cdot \text{V1m} \cdot \sin(\theta_{20} - \theta_{11}) + \text{Im}(Y_{bus_{1,2}}) \cdot V_{2m0} \cdot \text{V3m0} \cdot \sin(\theta_{20} - \theta_{30}) \\
J_{Q28_3} & := -\text{Im}(Y_{bus_{1,2}}) \cdot V_{2m0} \cdot \text{V3m0} \cdot \sin(\theta_{20} - \theta_{30}) \\
J_{Q38_2} & := -\text{Im}(Y_{bus_{2,1}}) \cdot V_{3m0} \cdot \text{V2m0} \cdot \sin(\theta_{30} - \theta_{20}) \\
J_{Q38_3} & := \text{Im}(Y_{bus_{2,0}}) \cdot V_{3m0} \cdot \text{V1m} \cdot \sin(\theta_{30} - \theta_{11}) + \text{Im}(Y_{bus_{2,1}}) \cdot V_{3m0} \cdot \text{V2m0} \cdot \sin(\theta_{30} - \theta_{20})
\end{align*}
\]

J22 submatrix
\[
\begin{align*}
J_{Q2V2_0} & := -\text{Im}(Y_{bus_{1,0}}) \cdot \text{V1m} \cdot \cos(\theta_{20} - \theta_{11}) - 2 \cdot V_{2m0} \cdot \text{Im}(Y_{bus_{1,1}}) - \text{Im}(Y_{bus_{1,2}}) \cdot V_{3m0} \cdot \cos(\theta_{20} - \theta_{30}) \\
J_{Q2V3_0} & := -\text{Im}(Y_{bus_{1,2}}) \cdot V_{2m0} \cdot \cos(\theta_{20} - \theta_{30}) \\
J_{Q3V2_0} & := -\text{Im}(Y_{bus_{2,1}}) \cdot V_{3m0} \cdot \cos(\theta_{30} - \theta_{20}) \\
J_{Q3V3_0} & := -\text{Im}(Y_{bus_{2,0}}) \cdot \text{V1m} \cdot \cos(\theta_{30} - \theta_{11}) - \text{Im}(Y_{bus_{2,1}}) \cdot V_{2m0} \cdot \cos(\theta_{30} - \theta_{20}) - 2 \cdot V_{3m0} \cdot \text{Im}(Y_{bus_{2,2}})
\end{align*}
\]
\[ J_0 = \begin{pmatrix} JP2\theta\_2 \_0 & JP2\theta\_3 \_0 & JP2Vm2\_0 & JP2Vm3\_0 \\ JP3\theta\_2 \_0 & JP3\theta\_3 \_0 & JP3Vm2\_0 & JP3Vm3\_0 \\ JQ2\theta\_2 \_0 & JQ2\theta\_3 \_0 & JQ2Vm2\_0 & JQ2Vm3\_0 \\ JQ3\theta\_2 \_0 & JQ3\theta\_3 \_0 & JQ3Vm2\_0 & JQ3Vm3\_0 \end{pmatrix} \]

\[ J_0 = \begin{pmatrix} 20 & -10 & 0 & 0 \\ -10 & 20 & 0 & 0 \\ 0 & 0 & 19.96 & -10 \\ 0 & 0 & -10 & 19.96 \end{pmatrix} \]

Now solve for \( \Delta x \) (note, I'm using the built-in matrix inverse, but for a large case, one would use LU factorization)

\[ \Delta x_1 := J_0^{-1} \cdot \Delta H_0 \]

\[ \Delta x_1 = \begin{pmatrix} 0 \\ -0.05 \\ 1.941 \times 10^{-3} \\ -0.048 \end{pmatrix} \]

\[ \theta_{21} := \theta_2 + \Delta x_{10} \]

\[ \theta_{31} := \theta_3 + \Delta x_{11} \]

\[ V_{2m1} := V_{2m0} + \Delta x_{12} \]

\[ V_{3m1} := V_{3m0} + \Delta x_{13} \]

\[ \theta_{21} = 0 \text{- deg} \]

\[ \theta_{31} = -2.865 \text{- deg} \]

\[ V_{2m1} = 1.002 \]

\[ V_{3m1} = 0.952 \]

Power flow equations using result of iteration 1:

\[ P_{21} := V_{2m1} \cdot V_{1m} \cdot \text{Im}(Y_{bus\_1,0}) \cdot \sin(\theta_{21} - \theta_1) + V_{2m1} \cdot V_{3m1} \cdot \text{Im}(Y_{bus\_1,2}) \cdot \sin(\theta_{21} - \theta_3) \]

\[ P_{31} := V_{3m1} \cdot V_{1m} \cdot \text{Im}(Y_{bus\_2,0}) \cdot \sin(\theta_{31} - \theta_1) + V_{3m1} \cdot V_{2m1} \cdot \text{Im}(Y_{bus\_2,1}) \cdot \sin(\theta_{31} - \theta_2) \]

\[ Q_{21} := -V_{2m1} \cdot V_{1m} \cdot \text{Im}(Y_{bus\_1,0}) \cdot \cos(\theta_{21} - \theta_1) - V_{2m1}^2 \cdot \text{Im}(Y_{bus\_1,1}) - V_{2m1} \cdot V_{3m1} \cdot \text{Im}(Y_{bus\_1,2}) \cdot \cos(\theta_{21} - \theta_3) \]

\[ Q_{31} := -V_{3m1} \cdot V_{1m} \cdot \text{Im}(Y_{bus\_2,0}) \cdot \cos(\theta_{31} - \theta_1) - V_{3m1} \cdot V_{2m1} \cdot \text{Im}(Y_{bus\_2,1}) \cdot \cos(\theta_{31} - \theta_2) - V_{3m1}^2 \cdot \text{Im}(Y_{bus\_2,2}) \]
Updated Mismatch Vector:
\[
\Delta H_1 := \begin{pmatrix}
P_2 - P_{21} \\
P_3 - P_{31} \\
Q_2 - Q_{21} \\
Q_3 - Q_{31}
\end{pmatrix} \quad \Delta H_1 = \begin{pmatrix}
0.023 \\
-0.048 \\
-0.013 \\
-0.071
\end{pmatrix}
\]

Use a "1-norm"
\[
\text{norm}_1 := \begin{cases}
\text{out} \leftarrow 0 \\
\text{for } x \in 0..3 \\
\text{out} \leftarrow \text{out} + |\Delta H_1|_1
\end{cases}
\]
\[
\text{norm}_1 = 0.155 \quad \text{good improvement}
\]

Jacobian Terms

**J11 submatrix**

\[JP_{21}\_1 := \text{Im}(Y_{bus_{1,0}}) \cdot V_{2m1} \cdot V_{1m} \cdot \cos(\theta_{21} - \theta_{31}) + \text{Im}(Y_{bus_{1,2}}) \cdot V_{2m1} \cdot V_{3m1} \cdot \cos(\theta_{21} - \theta_{31})\]

\[JP_{23}\_1 := -\text{Im}(Y_{bus_{1,2}}) \cdot V_{2m1} \cdot V_{3m1} \cdot \cos(\theta_{21} - \theta_{31})\]

\[JP_{32}\_1 := -\text{Im}(Y_{bus_{2,1}}) \cdot V_{3m1} \cdot V_{2m1} \cdot \cos(\theta_{31} - \theta_{21})\]

\[JP_{33}\_1 := \text{Im}(Y_{bus_{2,0}}) \cdot V_{3m1} \cdot V_{1m} \cdot \cos(\theta_{31} - \theta_{21}) + \text{Im}(Y_{bus_{2,1}}) \cdot V_{3m1} \cdot V_{2m1} \cdot \cos(\theta_{31} - \theta_{21})\]

**J12 submatrix**

\[JP_{2V_m2}\_1 := \text{Im}(Y_{bus_{1,0}}) \cdot V_{1m} \cdot \sin(\theta_{21} - \theta_{31}) + \text{Im}(Y_{bus_{1,2}}) \cdot V_{3m1} \cdot \sin(\theta_{21} - \theta_{31})\]

\[JP_{2V_m3}\_1 := \text{Im}(Y_{bus_{1,2}}) \cdot V_{2m1} \cdot \sin(\theta_{21} - \theta_{31})\]

\[JP_{3V_m2}\_1 := \text{Im}(Y_{bus_{2,1}}) \cdot V_{3m1} \cdot \sin(\theta_{31} - \theta_{21})\]

\[JP_{3V_m3}\_1 := \text{Im}(Y_{bus_{2,0}}) \cdot V_{1m} \cdot \sin(\theta_{31} - \theta_{21}) + \text{Im}(Y_{bus_{2,1}}) \cdot V_{2m1} \cdot \sin(\theta_{31} - \theta_{21})\]

**J21 submatrix**

\[JP_{2Q_2}\_1 := \text{Im}(Y_{bus_{1,0}}) \cdot V_{2m1} \cdot V_{1m} \cdot \sin(\theta_{21} - \theta_{31}) + \text{Im}(Y_{bus_{1,2}}) \cdot V_{2m1} \cdot V_{3m1} \cdot \sin(\theta_{21} - \theta_{31})\]

\[JP_{2Q_3}\_1 := -\text{Im}(Y_{bus_{1,2}}) \cdot V_{2m1} \cdot V_{3m1} \cdot \sin(\theta_{21} - \theta_{31})\]

\[JP_{3Q_2}\_1 := -\text{Im}(Y_{bus_{2,1}}) \cdot V_{3m1} \cdot V_{2m1} \cdot \sin(\theta_{31} - \theta_{21})\]

\[JP_{3Q_3}\_1 := \text{Im}(Y_{bus_{2,0}}) \cdot V_{3m1} \cdot V_{1m} \cdot \sin(\theta_{31} - \theta_{21}) + \text{Im}(Y_{bus_{2,1}}) \cdot V_{3m1} \cdot V_{2m1} \cdot \sin(\theta_{31} - \theta_{21})\]
J22 submatrix

\[ JQ2Vm2_1 := -\text{Im}(Y_{bus_{1,0}}) V_{1m} \cos(\theta_{21} - \theta_{1}) - 2 V_{2m1} \text{Im}(Y_{bus_{1,1}}) - \text{Im}(Y_{bus_{1,2}}) V_{3m1} \cos(\theta_{21} - \theta_{31}) \]

\[ JQ2Vm3_1 := -\text{Im}(Y_{bus_{1,2}}) V_{2m1} \cos(\theta_{21} - \theta_{31}) \]

\[ JQ3Vm2_1 := -\text{Im}(Y_{bus_{2,1}}) V_{3m1} \cos(\theta_{31} - \theta_{21}) \]

\[ JQ3Vm3_1 := -\text{Im}(Y_{bus_{2,0}}) V_{1m} \cos(\theta_{31} - \theta_{1}) - \text{Im}(Y_{bus_{2,1}}) V_{2m1} \cos(\theta_{31} - \theta_{21}) - 2 V_{3m1} \text{Im}(Y_{bus_{2,2}}) \]

\[ J_{-1} := \begin{bmatrix} 
  JP202_1 & JP203_1 & JP2Vm2_1 & JP2Vm3_1 \\
  JP302_1 & JP303_1 & JP3Vm2_1 & JP3Vm3_1 \\
  JQ202_1 & JQ203_1 & JQ2Vm2_1 & JQ2Vm3_1 \\
  JQ302_1 & JQ303_1 & JQ3Vm2_1 & JQ3Vm3_1 
\end{bmatrix} \]

\[ J_{-1} = \begin{bmatrix} 
  19.545 & -9.525 & 0.476 & 0.501 \\
  -9.525 & 19.032 & -0.476 & -1.001 \\
  0.477 & -0.477 & 20.531 & -10.007 \\
  0.477 & -0.952 & -9.507 & 18.043 
\end{bmatrix} \]

Now solve for \( \Delta x \)

\[ \Delta x_2 := J_{-1}^{-1} \Delta H_1 \]

\[ \Delta x_2 = \begin{bmatrix} 
  2.542 \times 10^{-5} \\
  -2.894 \times 10^{-3} \\
  -3.621 \times 10^{-3} \\
  -5.998 \times 10^{-3} 
\end{bmatrix} \]

\[ \theta_{22} := \theta_{21} + \Delta x_{20} \]

\[ \theta_{32} := \theta_{31} + \Delta x_{21} \]

\[ V_{2m2} := V_{2m1} + \Delta x_{22} \]

\[ V_{3m2} := V_{3m1} + \Delta x_{23} \]

\[ \theta_{22} = 1.457 \times 10^{-2} \text{ deg} \]

\[ \theta_{32} = -3.031 \text{ deg} \]

\[ V_{2m2} = 0.998 \]

\[ V_{3m2} = 0.946 \]
Power flow equations using result of iteration 2:

\[ P_{22} := V_{2m2} \cdot V_{1m} \cdot \text{Im}(Y_{bus_{1,0}}) \cdot \sin(\theta_{22} - \theta_{1}) + V_{2m2} \cdot V_{3m2} \cdot \text{Im}(Y_{bus_{1,2}}) \cdot \sin(\theta_{22} - \theta_{32}) \]

\[ P_{32} := V_{3m2} \cdot V_{1m} \cdot \text{Im}(Y_{bus_{2,0}}) \cdot \sin(\theta_{32} - \theta_{1}) + V_{3m2} \cdot V_{2m2} \cdot \text{Im}(Y_{bus_{2,1}}) \cdot \sin(\theta_{32} - \theta_{22}) \]

\[ Q_{22} := -V_{2m2} \cdot V_{1m} \cdot \text{Im}(Y_{bus_{1,0}}) \cdot \cos(\theta_{22} - \theta_{1}) - V_{2m2} \cdot V_{3m2} \cdot \text{Im}(Y_{bus_{1,1}}) - V_{2m2} \cdot V_{3m2} \cdot \text{Im}(Y_{bus_{1,2}}) \cdot \cos(\theta_{22} - \theta_{32}) \]

\[ Q_{32} := -V_{3m2} \cdot V_{1m} \cdot \text{Im}(Y_{bus_{2,0}}) \cdot \cos(\theta_{32} - \theta_{1}) - V_{3m2} \cdot V_{2m2} \cdot \text{Im}(Y_{bus_{2,1}}) \cdot \cos(\theta_{32} - \theta_{22}) - V_{3m2} \cdot V_{2m2} \cdot \text{Im}(Y_{bus_{2,2}}) \]

Updated Mismatch Vector:

\[ \Delta H_2 := \begin{pmatrix} P_2 - P_{22} \\ P_3 - P_{32} \\ Q_2 - Q_{22} \\ Q_3 - Q_{32} \end{pmatrix} \]

\[ \Delta H_2 = \begin{pmatrix} 2.672 \times 10^{-4} \\ -4.417 \times 10^{-4} \\ -7.148 \times 10^{-5} \\ -5.592 \times 10^{-4} \end{pmatrix} \]

Use a "1-norm"

\[ \text{norm}_1 := \begin{cases} \text{out} \leftarrow 0 & \text{for } x \in 0..3 \\ \text{out} \leftarrow \text{out} + |\Delta H_2_x| & \end{cases} \]

\[ \text{norm}_1 = 1.34 \times 10^{-3} \]

close to a desirable tolerance

Find P and Q at slack bus:

\[ P_1 := V_{1m} \cdot V_{2m2} \cdot \text{Im}(Y_{bus_{0,1}}) \cdot \sin(\theta_1 - \theta_{22}) + V_{1m} \cdot V_{3m2} \cdot \text{Im}(Y_{bus_{0,2}}) \cdot \sin(\theta_1 - \theta_{32}) \]

\[ P_1 = 0.5 \quad P_1 + P_2 = 1 \]

\[ Q_1 := -V_{1m}^2 \cdot \text{Im}(Y_{bus_{0,0}}) - V_{2m2} \cdot \text{Im}(Y_{bus_{0,1}}) \cdot \cos(\theta_1 - \theta_{22}) - V_{1m} \cdot V_{3m2} \cdot \text{Im}(Y_{bus_{0,2}}) \cdot \cos(\theta_1 - \theta_{32}) \]

\[ Q_1 = 0.551 \quad Q_1 + Q_{22} = 1.051 \]
Newton-Raphson Example 2:

Now suppose that we were given that the voltage at bus 2 is known (voltage regulated bus):

\[ V_{2m} := 1.02 \]

Now \( V_{2m} \) is no longer an unknown, which reduces the size of the problem to solve.

Power flow equations using initial guess:

\[ P_{20} := V_{2m} \cdot V_{1m} \cdot \text{Im}(Y_{bus_{1,0}}) \cdot \sin(\theta_{20} - \theta_1) + V_{2m} \cdot V_{3m} \cdot \text{Im}(Y_{bus_{1,2}}) \cdot \sin(\theta_{20} - \theta_{30}) \]

\[ P_{30} := V_{3m} \cdot V_{1m} \cdot \text{Im}(Y_{bus_{2,0}}) \cdot \sin(\theta_{30} - \theta_1) + V_{3m} \cdot V_{2m} \cdot \text{Im}(Y_{bus_{2,1}}) \cdot \sin(\theta_{30} - \theta_{20}) \]

Q2 equation is no longer needed here.

\[ Q_{30} := -V_{3m} \cdot V_{1m} \cdot \text{Im}(Y_{bus_{2,0}}) \cdot \cos(\theta_{30} - \theta_1) - V_{3m} \cdot V_{2m} \cdot \text{Im}(Y_{bus_{2,1}}) \cdot \cos(\theta_{30} - \theta_{20}) - V_{3m}^2 \cdot \text{Im}(Y_{bus_{2,2}}) \]

Initial Mismatch Vector:

\[ \Delta H_0 = \begin{pmatrix} P_2 - P_{20} \\ P_3 - P_{30} \\ Q_3 - Q_{30} \end{pmatrix} \]

\[ \Delta H_0 = \begin{pmatrix} 0.5 \\ -1 \\ -0.78 \end{pmatrix} \]

Use a "1-norm"

\[ \text{norm}_1 := \begin{cases} \text{out} \leftarrow 0 \\ \text{for} \ x \in 0..2 \\ \text{out} \leftarrow \text{out} + |\Delta H_0| \end{cases} \]

\[ \text{norm}_1 = 2.28 \quad \text{well out of tolerance} \]

Jacobian Terms

J11 submatrix

\[ J_{11_{2,0}} := \text{Im}(Y_{bus_{1,0}}) \cdot V_{2m} \cdot V_{1m} \cdot \cos(\theta_{20} - \theta_1) + \text{Im}(Y_{bus_{1,2}}) \cdot V_{2m} \cdot V_{3m} \cdot \cos(\theta_{20} - \theta_{30}) \]

\[ J_{11_{3,0}} := -\text{Im}(Y_{bus_{1,2}}) \cdot V_{2m} \cdot V_{3m} \cdot \cos(\theta_{20} - \theta_{30}) \]

\[ J_{11_{2,1}} := -\text{Im}(Y_{bus_{2,1}}) \cdot V_{3m} \cdot V_{2m} \cdot \cos(\theta_{30} - \theta_{20}) \]

\[ J_{11_{3,1}} := \text{Im}(Y_{bus_{2,0}}) \cdot V_{3m} \cdot V_{1m} \cdot \cos(\theta_{30} - \theta_1) + \text{Im}(Y_{bus_{2,1}}) \cdot V_{3m} \cdot V_{2m} \cdot \cos(\theta_{30} - \theta_{20}) \]
J12 submatrix (no longer have partials with respect to V2m)

\[ JP2V_m := \text{Im}(Ybus_{1,2}) \cdot V2m \cdot \sin(\theta 20 - \theta 30) \]

\[ JP3V_m := \text{Im}(Ybus_{2,0}) \cdot V1m \cdot \sin(\theta 30 - \theta 1) + \text{Im}(Ybus_{2,1}) \cdot V2m \cdot \sin(\theta 30 - \theta 20) \]

J21 submatrix (no longer have Q2 terms)

\[ JQ3\theta_2 := -\text{Im}(Ybus_{2,1}) \cdot V3m_0 \cdot V2m \cdot \sin(\theta 30 - \theta 20) \]

\[ JQ3\theta_3 := \text{Im}(Ybus_{2,0}) \cdot V3m_0 \cdot V1m \cdot \sin(\theta 30 - \theta 1) + \text{Im}(Ybus_{2,1}) \cdot V3m_0 \cdot V2m \cdot \sin(\theta 30 - \theta 20) \]

J22 submatrix

\[ JQ2V_m := -\text{Im}(Ybus_{1,2}) \cdot V2m \cdot \cos(\theta 20 - \theta 30) \]

\[ JQ3V_m := -\text{Im}(Ybus_{2,0}) \cdot V1m \cdot \cos(\theta 30 - \theta 1) - \text{Im}(Ybus_{2,1}) \cdot V2m \cdot \cos(\theta 30 - \theta 20) - 2 \cdot V3m_0 \cdot \text{Im}(Ybus_{2,2}) \]

\[ J_{0} := \begin{pmatrix} JP2\theta_2 & JP3\theta_2 & JP2V_m \\ JP3\theta_2 & JP3\theta_3 & JP3V_m \\ JQ3\theta_2 & JQ3\theta_3 & JQ3V_m \end{pmatrix} \]

\[ J_{0} = \begin{pmatrix} 20.4 & -10.2 & 0 \\ -10.2 & 20.2 & 0 \\ 0 & 0 & 19.76 \end{pmatrix} \]

Now solve for \( \Delta x \) (note, I'm using the built-in matrix inverse, but for a large case, we would use LU factorization)

\[ \Delta x \] := \( J_{0}^{-1} \cdot \Delta H_0 \)

\[ \Delta x = \begin{pmatrix} -3.246 \times 10^{-4} \\ -0.05 \\ -0.039 \end{pmatrix} \]

\[ \theta 21 := \theta 20 + \Delta x_1 \]

\[ \theta 31 := \theta 30 + \Delta x_1 \]

\[ V3m_1 := V3m_0 + \Delta x_2 \]

\[ V3m_1 = 0.961 \]
Power flow equations using result of iteration 1:

\[ P_{21} := V_{2m} \cdot V_{1m} \cdot \text{Im}(Y_{bus_{1,0}}) \cdot \sin(\theta_{21} - \theta_{1}) + V_{2m} \cdot V_{3m} \cdot \text{Im}(Y_{bus_{1,2}}) \cdot \sin(\theta_{21} - \theta_{31}) \]

\[ P_{31} := V_{3m} \cdot V_{1m} \cdot \text{Im}(Y_{bus_{2,0}}) \cdot \sin(\theta_{31} - \theta_{1}) + V_{3m} \cdot V_{2m} \cdot \text{Im}(Y_{bus_{2,1}}) \cdot \sin(\theta_{31} - \theta_{21}) \]

\[ Q_{31} := -V_{3m} \cdot V_{1m} \cdot \text{Im}(Y_{bus_{2,0}}) \cdot \cos(\theta_{31} - \theta_{1}) - V_{3m} \cdot V_{2m} \cdot \text{Im}(Y_{bus_{2,1}}) \cdot \cos(\theta_{31} - \theta_{21}) - V_{3m}^2 \cdot \text{Im}(Y_{bus_{2,2}}) \]

Updated Mismatch Vector:

\[
\Delta H_1 := \begin{pmatrix} P_2 - P_{21} \\ P_3 - P_{31} \\ Q_3 - Q_{31} \end{pmatrix} \quad \Delta H_1 = \begin{pmatrix} 0.02 \\ -0.04 \\ -0.055 \end{pmatrix} \]

Use a "1-norm"

\[
\text{norm}_1 := \begin{cases} \text{out} & \text{for } x \in 0..2 \\ \text{out} \leftarrow \text{out} + |\Delta H_1| \end{cases} \]

\[
\text{norm}_1 = 0.115
\]

Jacobian Terms

J11 submatrix

\[ J_{P202_1} := \text{Im}(Y_{bus_{1,0}}) \cdot V_{2m} \cdot V_{1m} \cdot \cos(\theta_{21} - \theta_{1}) + \text{Im}(Y_{bus_{1,2}}) \cdot V_{2m} \cdot V_{3m} \cdot \cos(\theta_{21} - \theta_{31}) \]

\[ J_{P202_1} := -\text{Im}(Y_{bus_{1,2}}) \cdot V_{2m} \cdot V_{3m} \cdot \cos(\theta_{21} - \theta_{1}) \]

\[ J_{P302_1} := -\text{Im}(Y_{bus_{2,1}}) \cdot V_{3m} \cdot V_{2m} \cdot \cos(\theta_{31} - \theta_{21}) \]

\[ J_{P302_1} := \text{Im}(Y_{bus_{2,0}}) \cdot V_{3m} \cdot V_{1m} \cdot \cos(\theta_{31} - \theta_{1}) + \text{Im}(Y_{bus_{2,1}}) \cdot V_{3m} \cdot V_{2m} \cdot \cos(\theta_{31} - \theta_{21}) \]

J12 submatrix

\[ J_{P2V_{m3,1}} := \text{Im}(Y_{bus_{1,2}}) \cdot V_{2m} \cdot \sin(\theta_{21} - \theta_{31}) \]

\[ J_{P3V_{m3,1}} := \text{Im}(Y_{bus_{2,0}}) \cdot V_{1m} \cdot \sin(\theta_{31} - \theta_{1}) + \text{Im}(Y_{bus_{2,1}}) \cdot V_{2m} \cdot \sin(\theta_{31} - \theta_{21}) \]

J21 submatrix

\[ J_{Q302_1} := -\text{Im}(Y_{bus_{2,1}}) \cdot V_{3m} \cdot V_{2m} \cdot \sin(\theta_{31} - \theta_{21}) \]

\[ J_{Q302_1} := \text{Im}(Y_{bus_{2,0}}) \cdot V_{3m} \cdot V_{1m} \cdot \sin(\theta_{31} - \theta_{1}) + \text{Im}(Y_{bus_{2,1}}) \cdot V_{3m} \cdot V_{2m} \cdot \sin(\theta_{31} - \theta_{21}) \]

J22 submatrix

\[ J_{Q3V_{m3,1}} := -\text{Im}(Y_{bus_{2,0}}) \cdot V_{1m} \cdot \cos(\theta_{31} - \theta_{1}) - \text{Im}(Y_{bus_{2,1}}) \cdot V_{2m} \cdot \cos(\theta_{31} - \theta_{21}) - 2 \cdot V_{3m} \cdot \text{Im}(Y_{bus_{2,2}}) \]