However, for a rotating machine, the number of turns coupled by the flux will also vary since the windings are distributed.

So the Faraday's Law equation becomes:

\[ e = \frac{d}{dt} \lambda = N \frac{d}{dt} \phi(t) + \phi \frac{d}{dt} N(t) \]

The machine case is further complicated by the coupling between phases, so \( \lambda_a \) will be impacted by the currents in the other phases and in the field circuit.

Recall that when we discussed magnetic circuits, we defined a term called "Reluctance"

\[ L = \frac{N^2}{\text{Rel}} \]

We can use \( L \) to related the flux linkages to the currents in each coupled circuit

\[ \lambda_a = L_{aa}i_a(t) + L_{ab}i_b(t) + L_{ac}i_c(t) + L_{aF}i_f \]

- \( L_{aa} \) is the self inductance of phase A
- \( L_{ab} \) is mutual inductance between phases A and B
- \( L_{ac} \) is mutual inductance between phases A and C
- \( L_{aF} \) is mutual inductance between phases A the field winding \( F \)

We will find later that each of these inductances each have a constant part and a part that varies with time as the rotor turns.

As a first approximation, we can break the \( L_{aa} \) into:

\[ L_{aa} = L_{aa0} + L_{al} \]

where:

\[ L_{aa0} = \frac{N_s^2}{2 \cdot \text{Relag}} \]

\( N_s = \text{Stator}\_\text{turns} \)

\( L_{al} = \text{Leakage} \)

\( \text{Relag} = \text{Reluctance across the air gap} \)
Salient Pole Machine Equations

**Inductance Equations:**

Direct axis inductance of phase "s" (round rotor term)

\[ L_{ss} = h \cdot k \cdot N_s^2 \]

or we could say: \[ L_{ss} = \frac{N_s^2}{2 \cdot R_{lag}} \]

Saliency adjustment term:

\[ L_{\Delta} = \frac{\Delta h}{2} \cdot k \cdot N_s^2 \]

Coupling to rotor:

\[ L_{sf} = \left( h + \frac{\Delta h}{2} \right) \cdot k \cdot N_s \cdot N_f \]

\[ L_m = \frac{3}{2} \cdot L_{ss} \]

Self inductances:

\[ L_{aa}(\theta_r) = (L_{ss} + L_{ls}) - L_{\Delta} \cdot \cos(2 \cdot \theta_r) \]

\[ L_{bb}(\theta_r) = (L_{ss} + L_{ls}) - L_{\Delta} \cdot \cos\left(2 \cdot \left(\theta_r - \frac{2\pi}{3}\right)\right) = (L_{ss} + L_{ls}) - L_{\Delta} \cdot \cos\left(2 \cdot \theta_r + \frac{2\pi}{3}\right) \]

\[ L_{cc}(\theta_r) = (L_{ss} + L_{ls}) - L_{\Delta} \cdot \cos\left(2 \cdot \left(\theta_r + \frac{2\pi}{3}\right)\right) = (L_{ss} + L_{ls}) - L_{\Delta} \cdot \cos\left(2 \cdot \theta_r - \frac{2\pi}{3}\right) \]

Stator to stator mutual inductances

\[ L_{ab}(\theta_r) = \frac{-L_{ss}}{2} - L_{\Delta} \cdot \cos\left(2 \cdot \theta_r - \frac{2\pi}{3}\right) \]

\[ L_{ac}(\theta_r) = \frac{-L_{ss}}{2} - L_{\Delta} \cdot \cos\left(2 \cdot \theta_r + \frac{2\pi}{3}\right) \]

\[ L_{bc}(\theta_r) = \frac{-L_{ss}}{2} - L_{\Delta} \cdot \cos(2 \cdot \theta_r) \]

all symmetric: \[ L_{ab} = L_{ba} \] and so on.
\[ \bar{\lambda}_{aarm} = \left( \frac{3}{2} \cdot L_{ss} + L_{ls} \right) \cdot \left( \cos(\theta_i) - j \cdot \sin(\theta_i) \right) - \frac{3}{2} \cdot L_{\Delta} \cdot |I_s| \cdot \left( \cos(\theta_i) + j \cdot \sin(\theta_i) \right) \]

\[ \bar{\lambda}_{aarm} = \left( \frac{3}{2} \cdot L_{ss} + L_{ls} - \frac{3}{2} \cdot L_{\Delta} \right) \cdot |I_s| \cdot \cos(\theta_i) - j \left( \frac{3}{2} \cdot L_{ss} + L_{ls} + \frac{3}{2} \cdot L_{\Delta} \right) \cdot |I_s| \cdot \sin(\theta_i) \]

Define quadrature and direct axis inductances:

\[ L_q = \frac{3}{2} \cdot L_{ss} + L_{ls} - \frac{3}{2} \cdot L_{\Delta} \]

\[ L_d = \frac{3}{2} \cdot L_{ss} + L_{ls} + \frac{3}{2} \cdot L_{\Delta} \]

Note the cause of the difference between \( L_d \) and \( L_q \)

Define quadrature and direct axis currents:

\[ \bar{i}_{aq} = |I_s| \cdot \cos(\theta_i) \cdot e^{j \cdot 0} \]

\[ \bar{i}_{ad} = -j \cdot |I_s| \cdot \sin(\theta_i) \cdot e^{j \cdot \frac{\pi}{2}} = |I_s| \cdot \sin(\theta_i) \cdot e^{-j \cdot \frac{\pi}{2}} \]

Therefore:

\[ \bar{\lambda}_{aarm} = L_q \cdot \bar{i}_{aq} + L_d \cdot \bar{i}_{ad} \]

\[ \bar{\lambda}_a = \bar{\lambda}_{af} + \bar{\lambda}_{aq} + \bar{\lambda}_{ad} = L_{sf} \cdot \bar{i}_f + L_q \cdot \bar{i}_{aq} + L_d \cdot \bar{i}_{ad} \]

Recall the angle of the field current:

\[ \bar{\lambda}_{aq} = L_q \cdot \bar{i}_{aq} \]

\[ \bar{\lambda}_{ad} = L_{sf} \cdot \bar{i}_f + L_d \cdot \bar{i}_{ad} \]

Back to the voltage equation:

\[ v_a(t) = -r_a \cdot i_a(t) - \frac{d}{dt} \lambda_a \]
Three Phase Short Circuit of a Synchronous Machine

\[ i_{as}(t) \approx \sqrt{2}|\hat{E}_a| \left[ \frac{1}{X_d} + \left( \frac{1}{X_d} - \frac{1}{X_d} \right) e^{-\frac{t}{T_d}} + \left( \frac{1}{X_q} - \frac{1}{X_q} \right) e^{-\frac{t}{T_q}} \right] \sin(\omega_e t + \alpha) \]

\[ -\sqrt{2}|\hat{E}_a| \frac{1}{2} \left( \frac{1}{X_d} + \frac{1}{X_q} \right) e^{-\frac{t}{T_d}} \sin(\alpha) \]

\[ -\sqrt{2}|\hat{E}_a| \frac{1}{2} \left( \frac{1}{X_d} - \frac{1}{X_q} \right) e^{-\frac{t}{2T_d}} \sin(2\omega_e t + \alpha) \]

<table>
<thead>
<tr>
<th>Name</th>
<th>Magnitude</th>
<th>Frequency</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steady</td>
<td>$E_a \frac{1}{X_d}$</td>
<td>Fundamental</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Transient</td>
<td>$E_a \left( \frac{1}{X_d} - \frac{1}{X_d} \right)$</td>
<td>Fundamental</td>
<td>$T'_d$</td>
</tr>
<tr>
<td>Subtransient</td>
<td>$E_a \left( \frac{1}{X_d} - \frac{1}{X_d} \right)$</td>
<td>Fundamental</td>
<td>$T''_d$</td>
</tr>
<tr>
<td>Asymmetrical</td>
<td>$\frac{E_a}{2} \left( \frac{1}{X_d} + \frac{1}{X_q} \right) \sin(\alpha)$</td>
<td>Zero</td>
<td>$T_a$</td>
</tr>
<tr>
<td>Second Harmonic</td>
<td>$\frac{E_a}{2} \left( \frac{1}{X_d} - \frac{1}{X_q} \right)$</td>
<td>Double Fundamental</td>
<td>$T_a$</td>
</tr>
</tbody>
</table>
- How do we use this in fault studies in phase domain?

\( \text{choose } \Rightarrow X \), based on time of interest
If you want maximum fault current, choose $X = X''$.

If interested in fault current when breakers are opening, also need to include dc offset term.

$\frac{2}{3}I_{AC} + \frac{1}{3}I = I$

The correct answer is $3X_i = x_1$. If interested to zone 2 or later, $3X_i = x_1$. Interruption of protection.
For fault studies

\[ X_2 \text{ is either value provided or } \frac{X_6'' + X_9''}{2} \]

\( R_2 \)

\( X_6 \) is based on data provided

\( \Rightarrow \) Grounding of the machine

Not time varying
Large generators - Ground

[Diagram of a power system with terminals A, B, and C, showing ground connection at neutral]

\[ R_{\text{ground}} \rightarrow \text{If there is a SLG at terminals } I_{4g} \leq 25 \text{ A (max)} \]

- much smaller

\[ 3R_G = X_{CO} \]
Fault analysis with generators

1. Determine time period of interest
2. Based on 1, choose $X_1$
3. Find prefault $|E_A|$ for the protection calculation
4. Plug into fault calc with $Z_x, Z_o$ etc.

For fault

Based on choice of $X_i$

$E_A' \leftrightarrow E_A''$

$E_i' \leftrightarrow E_i''$
• Voltage due to $B_s$ (armature reaction)

$$\frac{3}{2}L_{aa0}\frac{d}{dt}i_a(t)$$

Now define the **direct axis synchronous reactance**:

$$X_d = 2\cdot\pi \cdot 60 \text{ Hz} \left(\frac{3}{2}L_{aa0} + L_{al}\right)$$

Dominated by $L_{aa0}$ since leakage is small

So the voltage equation becomes:

$$v_a(t) = r_a'i_a(t) + L_s'\frac{d}{dt}i_a(t) + e_a(t)$$

Or in phasor form:

$$\bar{V}_a = \bar{r}_a\bar{I}_a + j\cdot X_{s}'\bar{I}_a + \bar{E}_a$$

Think back to dc machine, this implies current entering machine (motor operation)

Generator equation:

$$\bar{V}_a = \bar{E}_a - r\bar{I}_a - j\cdot X_s\bar{I}_a$$

Per Phase Equivalent Circuit (assumes Y connected):

• For a large machine it is generally possible to neglect $R_a$
• Normally the $X/R$ ratio is over 20
**Phasor Diagram of a Salient Pole Synchronous Generator**

- Define q-axis
  \[ a_l = V_a + R_a I_a + j X_q I_a \]
  \[ \theta_q = \text{arg}(a_l) \]

- Angle of the back EMF, \( E_a \)
  \[ E_a = V_{at} + R_a I_a + j X_d I_{ad} + j X_q I_{aq} \]
  \[ \theta_q = \delta_a = \text{arg}(E_a) \]

- Show that the angle of \( E_a \) is the same as the angle of \( a_l \)
  \[ E_a - a_l = \left( V_{at} + R_a I_a + j X_d I_{ad} + j X_q I_{aq} \right) - \left( V_a + R_a I_a + j X_q I_a \right) \]

- Which simplified to
  \[ E_a - a_l = \left( j X_d I_{ad} + j X_q I_{aq} \right) - \left( j X_q I_a \right) \]

- By definition we know that:
  \[ I_a = I_{ad} + I_{aq} \]

- Substituting
  \[ E_a - a_l = \left( j X_d I_{ad} + j X_q I_{aq} \right) - \left( j X_q I_a \right) \]

- Which simplified to:
  \[ E_a - a_l = I_{ad} \left( j X_d - j X_q \right) \]
• Then expressing in polar form

\[ E_a - a1 = \left[ \left| I_{ad} \right| \angle (\theta_q - 90\text{deg}) \right] \left[ \left( X_d - X_q \right) \angle (90\text{deg}) \right] \]

• Simplying the -90 degrees cancels the +90 degrees, and we see that the angle of \((E_a - a1)\) is the same as the angle of \(E_a\), therefore they are in phase with each other.

\[ E_a - a1 = \left[ \left| I_{ad} \right| \cdot \left( X_d - X_q \right) \right] \angle (\theta_q) \]

**Synchronous Machine Examples**

\( pu := 1 \)

Example 1: Unloaded synchronous Generator experiences a 3 phase fault at the high side of the station transformer

\[ X_{d}^{''} := 0.145 \text{pu} \]
\[ X_d := 0.240 \text{pu} \]
\[ X_{d} := 1.10 \text{pu} \]
\[ X_{d} := 0.1 \text{pu} \]
\[ V_{term} := 1.0 \text{pu} \]

Since unloaded:

\[ E_a' := V_{term} \]
\[ E_a := V_{term} \]
\[ E_a := V_{term} \]

\[ I_a'' := \frac{E_a''}{j \cdot X_d'' + j \cdot X_{tran}} \]
\[ I_a'' = -4.082i \cdot \text{pu} \]

\[ I_a' := \frac{E_a'}{j \cdot X_d' + j \cdot X_{tran}} \]
\[ I_a' = -2.941i \cdot \text{pu} \]

\[ I_{ss} := \frac{E_a}{j \cdot X_d + j \cdot X_{tran}} \]
\[ I_{ss} = -0.833i \cdot \text{pu} \]

\[ I_{dcoffsetmax} := \sqrt{2} \cdot \frac{E_a''}{j \cdot X_d'' + j \cdot X_{tran}} \]
\[ I_{dcoffsetmax} = -5.772i \cdot \text{pu} \]

Example 2: Repeat with generator operating at full load (pf = 0.8 lagging).

\[ \phi_{load} := \cos(0.8) \]
\[ \phi_{load} = 36.87 \cdot \text{deg} \]
\[ S_{\text{load}} := 1 \text{pu} \cdot e^{j \cdot \phi_{\text{load}}} \]

\[ I_{\text{load}} := \left( \frac{S_{\text{load}}}{V_{\text{term}}} \right) \]

\[ I_{\text{load}} = (0.8 - 0.6i) \cdot \text{pu} \quad |I_{\text{load}}| = 1 \cdot \text{pu} \]

\[ E''_{a2} := V_{\text{term}} + I_{\text{load}} \cdot j \cdot X''_d \quad |E''_{a2}| = 1.093 \cdot \text{pu} \quad \text{arg}(E''_{a2}) = 6.091 \cdot \text{deg} \]

\[ I_{\text{app}_2} := \frac{E''_{a2}}{j \cdot X''_d + j \cdot X_{\text{tran}}} \quad |I_{\text{app}_2}| = 4.462 \cdot \text{pu} \quad \text{arg}(I_{\text{app}_2}) = -83.909 \cdot \text{deg} \]

\[ \begin{align*}
E'_{a2} &= V_{\text{term}} + I_{\text{load}} \cdot j \cdot X'_d \\
I'_{a2} &= \frac{E'_{a2}}{j \cdot X'_d + j \cdot X_{\text{tran}}} \\
E_{a2} &= V_{\text{term}} + I_{\text{load}} \cdot j \cdot X_d \\
I_{a2} &= \frac{E_{a2}}{j \cdot X_d + j \cdot X_{\text{tran}}} \\
I_{\text{dcoffsetmax}_2} &= \sqrt{2 \cdot \frac{|E''_{a2}|}{j \cdot X''_d + j \cdot X_{\text{tran}}}} \\
I_{\text{dcoffsetmax}_2} &= 6.31 \cdot \text{pu}
\end{align*} \]

Example 3: A three phase fault occurs at the generator terminals, determine the symmetrical fault current at the instant of the fault, after 1/2 cycle, after 3 cycles, after 30 cycles and after 300 cycles. Assume generator is unloaded.

\[ T''_d := 0.035 \text{sec} \]

\[ T'_d := 0.730 \text{sec} \]

\[ t := 0 \text{sec} \]

\[ I_a(t) := E''_{a2} \left[ \frac{1}{X_d} + \left( \frac{1}{X'_d} - \frac{1}{X_d} \right) e^{\frac{-t}{T'_d}} + \left( \frac{1}{X''_d} - \frac{1}{X'_d} \right) e^{\frac{-t}{T''_d}} \right] \]
Instantaneous fault current: \[ I_a(t) = (7.497 + 0.8i) \cdot \text{pu} \]

After 1/2 cycle \[ t_1 := \frac{0.5}{60\text{Hz}} \quad \text{t}_1 = 8.333 \times 10^{-3} \text{s} \quad \left| I_a(t_1) \right| = 6.866 \cdot \text{pu} \]

After 3 cycles \[ t_2 := \frac{3}{60\text{Hz}} \quad \left| I_a(t_2) \right| = 5.034 \cdot \text{pu} \]

After 30 cycles \[ t_3 := \frac{30}{60\text{Hz}} \quad t_3 = 0.5 \text{s} \quad \left| I_a(t_3) \right| = 2.789 \cdot \text{pu} \]

- Should use \( E'_a \) for 30 cycles, not \( E''_a \)

After 300 cycles \[ t_4 := \frac{300}{60\text{Hz}} \quad \left| I_a(t_4) \right| = 0.998 \cdot \text{pu} \]

\[ I_{ss2} := \frac{E'_a}{X_d} \quad \left| I_{ss2} \right| = 1.708 \cdot \text{pu} \]

- Note the difference, should use \( E_a \) not \( E''_a \) now.

- Calculate momentary and interrupting ratings for generator breakers

\[ I_{dpMax} := \sqrt{3} \cdot \frac{E''_a}{X''_d} \quad I_{dpMax} = 11.945 \cdot \text{pu} \]

- Accounts dc offset as well as fundamental component

\[ I_{dpsym} := \frac{E''_a}{X''_d} \]

Momentary := 1.6 \cdot I_{dpsym} \quad \text{Momentary} = 11.034 \cdot \text{pu} \quad \text{at the minimum}

\[ M := 1.1 \quad \text{Assumes 5 cycle breakers (see Table 6.2)} \]

Interrupting := \( M \cdot I_{dpsym} \quad \text{Interrupting} = 7.586 \cdot \text{pu} \]

Determine these currents in Amps if:

1) The generator is rated at 13.2kV and 150 MVA

\[ \text{VB1} := 13.2\text{kV} \quad \text{MVA} := 1000\text{kW} \quad \text{SB1} := 150\text{MVA} \]