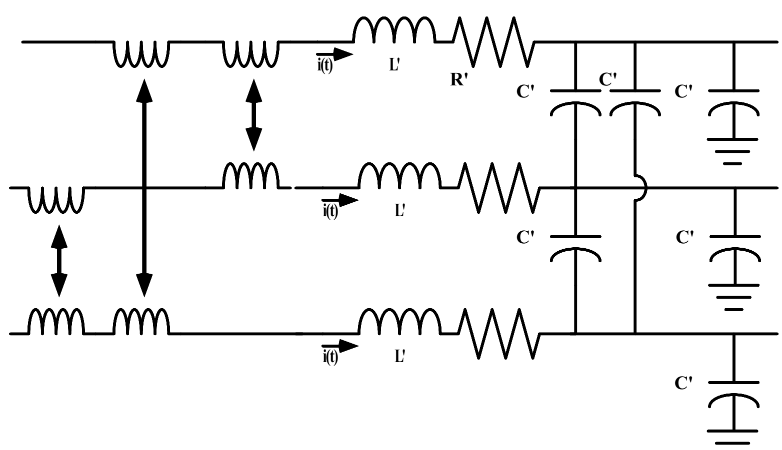
Multiphase Transmission Line Modeling

Basic Theory

The process of modeling lines with more than one conductor begins with the partial differential equations relating voltage and current on an incremental line segment with length Δx. The basic circuit elements for one of these conductors is shown below.



The partial differential equations representing this line segment in the limit as $\Delta x \to 0$ for the single conductor case now become matrix equations where:

$$-\frac{\partial e}{\partial x} = [R']i + [L']\frac{\partial i}{\partial t} \quad (1)$$

$$-\frac{\partial i}{\partial x} = [C']\frac{\partial e}{\partial t} \quad (2)$$

The analysis presented below can be applied to lines with any number of conductors, but we will continue the development for a line with 3 conductors, such that:

$$[R'] = \begin{bmatrix} R'_{aa} & R'_{ab} & R'_{ac} \\ R'_{ba} & R'_{bb} & R'_{bc} \\ R'_{ca} & R'_{cb} & R'_{cc} \end{bmatrix}$$

and

$$[L'] = \begin{bmatrix} L'_{aa} & L'_{ab} & L'_{ac} \\ L'_{ba} & L'_{bb} & L'_{bc} \\ L'_{ca} & L'_{cb} & L'_{cc} \end{bmatrix}$$

$$[C'] = \begin{bmatrix} C'_{aa} & C'_{ab} & C'_{ac} \\ C'_{ba} & C'_{bb} & C'_{bc} \\ C'_{ca} & C'_{cb} & C'_{cc} \end{bmatrix}$$

These matrices are generally symmetric. If the line is transposed, the elements on the main diagonal of each matrix are all equal, and the off-diagonal terms of each matrix are also equal.

If the system is excited at a particular frequency, ω, then the equations become:

$$-\frac{\partial E}{\partial x} = [Z']I \quad (3)$$

$$-\frac{\partial I}{\partial x} = [Y']E \quad (4)$$

where: $Z'_{ij} = R'_{ij} + j\omega L'_{ij}$ and $Y'_{ij} = j\omega C'_{ij}$. Note, that this will also result if a Fourier Transformation is applied to the equations.

Equations (3) and (4) can be re-written in the same manner used for the single phase case to give:

$$-\frac{\partial^2 E}{\partial x^2} = [Z'][Y']E \quad (5)$$

$$-\frac{\partial^2 I}{\partial x^2} = [Y'][Z']I \quad (6)$$

A modal transformation can be used to decouple these matrix equations. We can define two modal matrices, T_e and T_i, where T_e is the eigenvector matrix for $[Z'][Y']$ and T_i is the eigenvector matrix $[Y'][Z']$. The most effective way to calculate these is to use a dedicated eigenvalue routine, such as the EIG function in Matlab.

These are constant matrices, independent of time and position (x). We can then define vectors E_m and I_m to represent the variables transformed into the modal domain, such that:.

$$E = T_e E_m \quad (7)$$

$$I = T_i I_m \quad (8)$$

Next substitute E_m and I_m into equations (5) and (6) resulting in:

$$-[T_e]\frac{\partial^2 E_m}{\partial x^2} = [Z'][Y'][T_e]E_m \quad (9)$$

$$-[T_i]\frac{\partial^2 I_m}{\partial x^2} = [Y'][Z'][T_i]I_m \quad (10)$$

Which can be re-arranged as:

$$-\frac{\partial^2 E_m}{\partial x^2} = [T_e]^{-1}[Z'][Y'][T_e]E_m = \Lambda E_m \tag{11}$$

$$-\frac{\partial^2 I_m}{\partial x^2} = [T_i]^{-1}[Y'][Z'][T_i]I_m = \Lambda I_m \tag{12}$$

Notice that both $[Z'][Y']$ and $[Y'][Z']$ have the same eigenvalues. The theorem predicting this result is explained in many graduate level linear systems texts. The matrix Λ is a diagonal matrix of eigenvalues, resulting in m uncoupled equations, where m is the number of modes. Both voltage and current will have the same modes of propagation, as is demonstrated by having the same eigenvalues.

The transformation matrix T_e is not unique. Any matrix of the form $[D][T_e]$ is also a modal matrix. Where $[D] = d[I]$ where I is the identity matrix. In addition, T_e and T_i are related, as would be expected, since they lead to the same eigenvalues. We know that:

$$\Lambda = [T_e]^{-1}[Z'][Y'][T_e] \tag{13}$$

Then we can take the transpose

$$\Lambda^t = [[T_e]^{-1}[Z'][Y'][T_e]]^t \tag{14}$$

We know $\Lambda = \Lambda^t$ since Λ is diagonal. We can simplify this expression to be (recall, that $[Y']and[Z']$ are symmetric):

$$\Lambda = [T_e^t][Y'][Z'][T_e^t]^{-1} \tag{15}$$

So $[T_e^t]^{-1}$ is also a modal matrix for $[Y'][Z']$, which also has a modal matrix T_i defined by:

$$\Lambda = [T_i]^{-1}[Y'][Z'][T_i] \tag{16}$$

We can therefore conclude that:

$$[T_e][T_i^t] = [D] \tag{17}$$

where $[D] = d[I]$, and d could possibly equal one.

We can now diagonalize equations (3) and (4) as well.

$$-\frac{\partial E_m}{\partial x} = [T_e]^{-1}[Z'][T_i]I_m = [Z'_m]I_m \quad (18)$$

$$-\frac{\partial I_m}{\partial x} = [T_i]^{-1}[Y'][T_e]E_m = [Y'_m]E_m \quad (19)$$

We can show that both $[Z'_m]$ and $[Y'_m]$ are diagonal if we do the following:

$$\Lambda = [T_e]^{-1}[Z'][Y'][T_e] = ([T_e]^{-1}[Z']T_i)(T_i^{-1}[Y'][T_e]) = [Z'_m][Y'_m] \quad (20)$$

We can also see that:

$$\Lambda = [T_i]^{-1}[Y'][Z'][T_i] = ([T_i]^{-1}[Y']T_e)(T_e^{-1}[Z'][T_i]) = [Y'_m][Z'_m] \quad (21)$$

$\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_m) = [Z'_m][Y'_m] = [Y'_m][Z'_m]$ implies that $[Y'_m]$ and $[Z'_m]$ are diagonal. This holds for both the transposed and untransposed cases.

With the diagonalized form of the matrix differential equations in (20) and (21) the decoupled equations can now be solved as scalar equations using the methods discussed for single phase lines.

The resulting equations can be more easily solved if the resistance terms are lumped at the ends of the lines and lines are treated as lossless. Then for $E_m = [E_1, E_2, E_3, ..., E_n]^t$ and $I_m = [I_1, I_2, I_3, ..., I_n]^t$ we can find modal propagation velocities and characteristic impedances for representing the traveling waves on the lines as shown by:

$$\nu_0 = \frac{1}{\sqrt{L'_0 C'_0}} \quad and \quad Z_0 = \sqrt{\frac{L'_0}{C'_0}} \quad (22)$$

$$\nu_1 = \frac{1}{\sqrt{L'_1 C'_1}} \quad and \quad Z_1 = \sqrt{\frac{L'_1}{C'_1}} \quad (23)$$

$$\nu_2 = \frac{1}{\sqrt{L'_2 C'_2}} \quad and \quad Z_2 = \sqrt{\frac{L'_2}{C'_2}} \quad (24)$$

$$\cdots \quad (25)$$

$$\nu_n = \frac{1}{\sqrt{L'_n C'_n}} \quad and \quad Z_n = \sqrt{\frac{L'_n}{C'_n}} \quad (26)$$

However, solving the voltage equation using the modal transformation matrix T_e and the current equation using T_i will result in different values for the characteristic impedances. Therefore it will be necessary to include the transformation to get useful information. The propagation velocities will be the same in any case, since they come from the eigenvalues for the set of equations.

Uniformly Transposed Lines

The above analysis can be simplified if the transmission line is uniformly transposed. Then the matrices for $[L']$ and $[C']$ are simplified, with all of the diagonal elements equal and all of the off-diagonal elements equal, as shown below for a transposed 3 phase line with 3 conductors:

$$[L'] = \begin{bmatrix} L'_s & L'_{mu} & L'_{mu} \\ L'_{mu} & L'_s & L'_{mu} \\ L'_{mu} & L'_{mu} & L'_s \end{bmatrix}$$

and

$$[C'] = \begin{bmatrix} C'_s & C'_{mu} & C'_{mu} \\ C'_{mu} & C'_s & C'_{mu} \\ C'_{mu} & C'_{mu} & C'_s \end{bmatrix}$$

Then we'll find that $T_e = [Z'][Y'] = T_i = [Y'][Z']$ when we repeat the analysis presented in equations (3) through (8).

One possible transformation matrix that will accomplish this is the "Karrenbauer" transformation:

$$[T_i] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

and

$$[T_i]^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

However, the Clarke transformation (also called the $\alpha - \beta$ transform) is more commonly used in EMTP-like programs, where

$$[T_i] = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & \sqrt{2} & 0 \\ 1 & -\frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} \\ 1 & -\frac{1}{\sqrt{2}} & -\sqrt{\frac{3}{2}} \end{bmatrix}$$

and

$$[T_i]^{-1} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ \sqrt{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} \end{bmatrix}$$

Both of these transformations result in the same modal impedances and admittances as would result from applying the symmetrical components transformation in the 60 Hz phase domain. The advantage of the Clarke Transform is that $T_i^{-1} = T_i^t$, making the calculation of T_e or modal quantities easier. In this case we see that:

$$[Z'_m] = \begin{bmatrix} Z'_s + 2Z'_{mu} & 0 & 0 \\ 0 & Z'_s - Z'_{mu} & 0 \\ 0 & 0 & Z'_s - Z'_{mu} \end{bmatrix}$$

and

$$[Y'_m] = \begin{bmatrix} Y'_s + 2Y'_{mu} & 0 & 0 \\ 0 & Y'_s - Y'_{mu} & 0 \\ 0 & 0 & Y'_s - Y'_{mu} \end{bmatrix}$$

Then solving

$$-\frac{\partial E_m}{\partial x} = [Z'_m]I_m \tag{27}$$

$$-\frac{\partial I_m}{\partial x} = [Y'_m]E_m \tag{28}$$

where $E_m = [E_0, E_1, E_2]^t$ and $I_m = [I_0, I_1, I_2]^t$. shows that each set of three differential equations only has two unique equations. Modes 1 and 2 will have identical solutions. This is only true in the transposed case.

We can solve the differential equations for each modal voltage and current separately, and repeat the analysis used for single phase lines, only in this case, each single phase line represents a different mode. It is again expedient to use lossless line models, and lump the resistances at the ends of the lines.

In this case we'll propagation velocities and characteristic impedances for each of the modal voltages and currents, where:

$$\nu_0 = \frac{1}{\sqrt{L'_0 C'_0}} \quad and \quad Z_0 = \sqrt{\frac{L'_0}{C'_0}} \tag{29}$$

$$\nu_1 = \frac{1}{\sqrt{L'_1 C'_1}} \quad and \quad Z_1 = \sqrt{\frac{L'_1}{C'_1}} \tag{30}$$

$$\nu_2 = \nu_1 \qquad Z_1 = Z_2 \tag{31}$$

In this case the characteristic impedances are the same for the voltage and current equations if the Karrenbauer transformation is used.

EMTP Representation of Transmission Lines

The Electromagnetic Transients Program (EMTP) has the ability to use distributed parameter line models. The basic data card used for entering data into EMTP is a shown:

```
C Distributed parameter line.
C                                                   I I I
C                                                   L P P
C                                                   I U O
C                                                   N N S
C                                                   E C E
C                                                     H
C Bus1->Bus2->Bus3->Bus4-><---R'<----A<----B<--len 0 0 0<-------------------->0
```

Where “Bus1” and “Bus2” are the buses either end of the line. The fields for “Bus3” and “Bus4” are used for copying lines. The other fields are used for the following:

R’	series resistance per unit length (modal resistance)
A,B	Modal L’ in mH and C’ in μF (both per unit length) if ILINE =0
A,B	Modal Z_c in Ω and ν if ILINE = 1
A,B	Modal Z_c in Ω and τ in sec if ILINE = 2
len	Line length, units consistent for all quantities on same line
ILINE	Specifies quantities for A and B
IPUNCH	0 = lumped resistance model, 1 = distortionless model*
IPOSE	0 = transposed, n = number of phases in untransposed line

* A distortionless line is one where: $\frac{R'}{L'} = \frac{G'}{C'}$ where G' is the parasitic shunt conductance. G' tends to be quite small for overhead lines.

If the user chooses a distortionless line (IPUNCH=1) then EMTP assumes that R' represents the total losses coming from $I^2 * R_{series} + V^2 * G_{shunt}$. The losses are divided based on: $\frac{R'_{series}}{L'} = \frac{G'}{C'} = 0.5 * R'$.

The distortionless model will not change the steady-state results but will change the initial transient response. It should only be used for positive sequence cases if at all.

Notice that the quantities specified are the modal quantities, not the ABC domain quantities. A transposed line is solved using the Karrenbauer transformation, *so that transformation must be used to compute the modal quantities!* For an untransposed line the modal transformation matrix must also be given.

Sample single phase line:

```
C Distributed parameter line.
C Bus-->Bus-->Bus-->Bus--><---R'<---L'<---C'<--len 0 0 0                       0
-1BUS1  BUS12            0.0243 .9238 .0126 24.14 0 0 0                       0
```

The "-1" in the first two columns signifies the first conductor of a distributed parameter line model, in this case, the only conductor.

Sample transposed three phase line:

```
C Distributed parameter line.
C Bus-->Bus-->Bus-->Bus--><---R'<---L'<---C'<--len 0 0 0                       0
-1BUS1A BUS12A           0.3167 3.222.00787 24.14 0 0 0                       0       MODE 0
-2BUS1B BUS12B           0.0243 .9238 .0126 24.14 0 0 0                       0       MODE 1
-3BUS1C BUS12C                                                                0       MODE 2 = MODE 1
```

Notice that the modal quantities are specified for only 2 modes. This is because modes "1" and "2" are identical for a transposed three phase line, so it would be redundant to enter both.

Untransposed three phase line:

```
C Distributed parameter line.
C Bus-->Bus-->Bus-->Bus--><---R'<---L'<---C'<--len 0 0 0                       0
-1BUS1A BUS12A           0.3140 3.196.00793 24.14 0 0 3                       0
-2BUS1B BUS12B           0.0247 1.015 .0115 24.14 0 0 3                       0
-3BUS1C BUS12C           0.0239 .8288 .0137 24.14 0 0 3                       0
C --------Ti<---------Ti<---------Ti                                          0
  0.59521098 -0.70710678 -0.41240852                                          0
  0.00000000  0.00000000  0.00000000                                          0
C
  0.53985903  0.00000000  0.81230439                                          0
  0.00000000  0.00000000  0.00000000                                          0
C
  0.59521098  0.70710678 -0.41240852                                          0
  0.00000000  0.00000000  0.00000000                                          0
```

This line has a "3" in the field for IPOSE, stating that there are three untransposed conductors. There are now 3 unique sets of modal quantities in this case. The modal transformation matrix was also entered for this case. There are 2 rows for each entry. The first row is the real part of the matrix entry and the second row is the imaginary part.

Notice that this is assuming that $T_e T_i^T = d[I]$ and that $T_i^{-1} L' T_i = L'_m$.