Development of the Swing Equation

ANGLES

$\theta_m$ rotor angle in Mechanical degrees

$\theta_{sm}$ System angle in Mechanical degrees

$$\delta_m = \theta_m - \theta_{sm} \quad (1)$$

$$\frac{d\delta_m}{dt} = \frac{d\theta_m}{dt} - \frac{d\theta_{sm}}{dt} \quad (2)$$

$$\omega_r = \frac{d\delta_m}{dt} = \omega_m - \omega_{sm} \quad (3)$$

where

$$\omega_m = \frac{d\theta_m}{dt}$$

$$\omega_{sm} = \frac{d\theta_{sm}}{dt}$$

As an aside that will be useful later, solve Eq. 3 for $\omega_m$

$$\omega_m = \frac{d\delta_m}{dt} + \omega_{sm} = \omega_r + \omega_{sm} \quad (4)$$

Returning to the main flow. Take the derivative of Eq. 3

$$\frac{d\omega_r}{dt} = \frac{d^2\delta_m}{dt^2} = \frac{d\omega_m}{dt} - \frac{d\omega_{sm}}{dt} \quad (5)$$

Assume $\omega_{sm}$ is a constant.

$$\frac{d\omega_r}{dt} = \frac{d^2\delta_m}{dt^2} = \frac{d\omega_m}{dt} \quad (6)$$
MECHANICAL ROTATIONAL EQUATIONS

From mechanics

\[ J \frac{d \omega_m}{dt} + B \omega_m = \tau_a = \tau'_m - \tau_e \] (7)

where

- \( J \) is the combined inertia of the turbine, shaft, rotor, and excitor
- \( B \) is the total damping
- \( \tau_a \) is the accelerating torque
- \( \tau'_m \) is the prime mover torque (Mechanical)
- \( \tau_e \) is the electrical counter torque

Equation 7 can be rewritten using Eq. 6 and Eq. 4

\[ J \frac{d^2 \delta_m}{dt^2} + B \left( \frac{d \delta_m}{dt} + \omega_{sm} \right) = \tau'_m - \tau_e \] (8)

Using the distributive property of multiplication.

\[ J \frac{d^2 \delta_m}{dt^2} + B \frac{d \delta_m}{dt} + B \omega_{sm} = \tau'_m - \tau_e \] (9)

Multiply by \( \omega_m \)

\[ \omega_m J \frac{d^2 \delta_m}{dt^2} + \omega_m B \frac{d \delta_m}{dt} + \omega_m B \omega_{sm} = \omega_m \tau'_m - \omega_m \tau_e \] (10)

Rearrange terms

\[ \omega_m J \frac{d^2 \delta_m}{dt^2} + \omega_m B \frac{d \delta_m}{dt} + \omega_m \tau_e = \omega_m \tau'_m - \omega_m B \omega_{sm} \] (11)
Equation 11 can be rewritten in terms of the electrical power, $P_e$, and mechanical power, $P_m$.

$$\omega_m J \frac{d^2 \delta_m}{dt^2} + \omega_m B \frac{d \delta_m}{dt} + P_e = P_m$$

(12)

where

$$P_e = \omega_m \tau_e$$

$$P_m = \omega_m J_m - \omega_m B \omega_{sm}$$

From Eq. 4 and recognizing that $\frac{d \delta_m}{dt} \ll \omega_{sm}$

$$\omega_m = \frac{d \delta_m}{dt} + \omega_{sm} \approx \omega_{sm}$$

(13)

$$\omega_{sm} J \frac{d^2 \delta_m}{dt^2} + \omega_{sm} B \frac{d \delta_m}{dt} + P_e = P_m$$

(14)

For a round rotor machine

$$P_e = P_{max} \cos(\delta)$$

(15)

where

$$P_{max} = \frac{V_i E_g}{X_T}$$

$$\delta = \delta_m \left( \frac{N_p}{2} \right)$$

Multiply the first term by 1 in the form $2 \omega_{sm}/2 \omega_{sm}$

$$\frac{2 \omega_{sm}}{2 \omega_{sm}} \omega_{sm} J \frac{d^2 \delta_m}{dt^2} + \omega_{sm} B \frac{d \delta_m}{dt} + P_{max} \cos(\delta) = P_m$$

(16)

$$\frac{2 \omega_{sm}^2}{2 \omega_{sm}} \omega_{sm} J \frac{d^2 \delta_m}{dt^2} + \omega_{sm} B \frac{d \delta_m}{dt} + P_{max} \cos(\delta) = P_m$$

(17)
The initial Kinetic Energy stored in the inertia is:

\[ W_{KE} = \frac{\omega_{sm}^2}{2} J \]  

(18)

\[ \frac{2}{\omega_{sm}} W_{KE} \frac{d^2 \delta_m}{dt^2} + \omega_{sm} B \frac{d\delta_m}{dt} + P_{max} \cos(\delta) = P_m \]  

(19)

Multiply the first two terms by 1 in the form \( \frac{N_p/2}{N_p/2} \)

\[ \frac{2}{\omega_{e}} W_{KE} \frac{d^2 \delta}{dt^2} + \frac{\omega_{sm} B}{N_p/2} \frac{d\delta}{dt} + P_{max} \cos(\delta) = P_m \]  

(20)

where

\[ \omega_e = \frac{N_p}{2} \omega_{sm} \]

Dividing Eq. 20 by \( S_{base} \) yields:

\[ \frac{2}{\omega_{e} S_{base}} W_{KE} \frac{d^2 \delta}{dt^2} + \frac{\omega_{sm}}{(N_p/2) S_{base}} B \frac{d\delta}{dt} + P_{max} \frac{pu}{pu} \cos(\delta) = P_{mpu} \]  

(21)

\[ \frac{2}{\omega_{e}} H \frac{d^2 \delta}{dt^2} + D \frac{d\delta}{dt} + P_{max} \frac{pu}{pu} \cos(\delta) = P_{mpu} \]  

(22)

where

\[ H = \frac{W_{KE}}{S_{base}} \]

\[ D = \frac{\omega_{sm} B}{(N_p/2) S_{base}} \]
Defining $M = \frac{2H}{\omega_o}$ finally yields the per unit swing equation.

$$M \frac{d^2 \delta}{dt^2} + D \frac{d\delta}{dt} + P_{\text{max}_{\text{pu}}} \cos(\delta) = P_{\text{mpu}}$$ (23)

With the CB’s open, $P_{\text{max}} = 0$.

$$M \frac{d^2 \delta}{dt^2} + D \frac{d\delta}{dt} = P_{\text{mpu}}$$ (24)
The Equal Area Criteria (s25b handout) ignores damping; i.e., \( D = 0 \). With the CB’s open \( (P_e = 0) \),

\[
M \frac{d^2 \delta}{dt^2} = P_{mpu} \tag{25}
\]

\[
\alpha = \frac{d^2 \delta}{dt^2} = \frac{P_{mpu}}{M} \tag{26}
\]

\[
\alpha = \frac{\omega_c P_{mpu}}{2H} \tag{27}
\]

\[
\alpha = \frac{N_p/2 \omega_{sm} P_{mpu}}{2H} \cong \text{a constant} \tag{28}
\]

\[
\alpha = \frac{d \omega_r}{dt} \tag{29}
\]

\[
\omega_r = \alpha t + \omega_{r0} \tag{30}
\]

\[
\omega_{r0} = 0 \tag{31}
\]

\[
\omega_r = \alpha t \tag{32}
\]

\[
\frac{d \delta}{dt} = \omega_r = \alpha t \tag{33}
\]

\[
\int_{\delta_0}^{\delta} d\delta = \int_{0}^{t} \alpha dt \tag{34}
\]

\[
\delta - \delta_0 = \frac{1}{2} \alpha t^2 \tag{35}
\]

\[
\delta = \delta_0 + \frac{1}{2} \alpha t^2 \tag{36}
\]